Analysis of the Relaxation of Wall Electrons in a Plasma

G. Ecker, K.-U. Riemann, and A. Scholz

Institut für Theoretische Physik, Ruhr-Universität, Bochum

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A procedure is developed which renders the simultaneous description of angular and energy relaxation of wall electrons in a plasma possible. The plasma is assumed to be field free, allowance is made for elastic and inelastic collisions with neutrals, as well as for electron-electron encounters. The procedure is based on an expansion in terms of the eigenfunctions of a suitable part of the kinetic equation. In the zeroth order, the problem is reduced to a boundary value problem of the Sturm type, and subsequent solution of a singular integral equation by standard technique. To account for the rest of the kinetic equation, we construct a Green's function which forms the basis for an iteration process. The application of the procedure is illustrated with an example.

Introduction

The problem of relaxation of electrons emitted from a wall into a plasma is of interest in the field of gaseous electronics (e. g. thermionic devices, electrode phenomena) as well as in fusion technology. The reason is that the distribution function of the electrons and its relaxation from the wall into the plasma influences the composition of the plasma e. g. via the exciting and ionizing collisions of the electrons.

Previous investigations [1, 2] are restricted to the consideration of the energy relaxation. They neglect that region in front of the wall where angular relaxation occurs without essential change in the energy distribution. This procedure effectively subdivides the total relaxation process into two model regions and, as a consequence, inevitably introduces the uncertainties connected with the transition region between these model regions. In addition, it does not provide information about the angular relaxation.

In contrast, the present investigation aims to describe simultaneously the angular and energy relaxation, avoiding the above drawbacks. It may be anticipated that this task is not an easy one due to the fact that, in general, the angular distribution of the wall electrons is completely rearranged within a distance of a few mean free paths from the wall. This feature is reflected mathematically in the appearance of singular integral equations. Particular difficulties can also be expected from the inclusion of inelastic collisions and of the interaction of the wall electrons with the plasma electrons, since the angular and energy relaxation occur on the same scale.

Therefore it cannot be expected to provide in this paper the final solution of the full problem in the whole range. Our aim will be to develop a principal procedure suitable to deal with the problem in general. The ability of the method will be demonstrated with instructive examples. Further development and application of this method is in progress.

Reprint requests to Prof. Dr. G. Ecker, Institut für Theoretische Physik der Ruhr-Universität, *D-4630 Bochum*, Universitätsstr. 150, NB 7/152.

The System

The model which we will study in this paper is characterized by the following assumptions:

We consider a one dimensional device with a planar wall in contact with a plasma. In a distance L of many mean free paths from this wall a second boundary condition is prescribed either by the undisturbed properties of the plasma or by the properties of a second wall.

Our plasma is stationary.

The degree of ionization of the plasma is limited by the requirement that electron-electron collisions may be neglected as far as angular deflection is concerned but not far as energy exchange is concerned. This of course implies that electron-ion collisions may be neglected altogether.

In our analysis, field effects are presently neglected. This means in general that the Debye length of the system should be small compared with the mean free path of the electrons emitted from the wall.

We assume that the state of the plasma into which the wall electrons relax is known. Trivially this is justified if the effect of the wall electrons is so small as not to affect the actual plasma state. In

general, of course, a simultaneous solution of the relaxation process of the electrons and the plasma state is required.

Formulation of the Problem

The electron distribution function in the stationary and field-free-case is calculated from the kinetic equation of the electrons

$$\mu v \frac{\partial f}{\partial z} = \left(\frac{\delta f}{\delta t}\right)_{\mathrm{el}} + \left(\frac{\delta f}{\delta t}\right)_{\mathrm{in}} + \left(\frac{\delta f}{\delta t}\right)_{\mathrm{ee}} . \qquad (1)$$

Here, z designates the space-coordinate of the one dimensional problem, v the absolute value of the velocity and μ the cosine of the angle of the velocity with the z-axis.

The elastic interaction of the electrons with the neutral gas $[(\delta f/\delta t)_{\mathrm{el}}]$ is described by the Boltzmann collision term in the "Lorentz approximation" [3]

$$\left(\frac{\delta f}{\delta t}\right)_{\mathrm{el}} = \frac{n_0}{v^2}\frac{\partial}{\partial v}\left\{\frac{m}{m_0} q_1 v^4 \left(f + \frac{k T_0}{m v}\frac{\partial f}{\partial v}\right)\right\} - n_0 v \sum_{l=1}^{\infty} q_l(v) f^l(z,v) P_l(\mu) \qquad (2)$$

with the abbreviations

$$f^l(z,v) = \frac{2l+1}{2}\int_{-1}^{1} f(z,v,\mu) P_l(\mu)\,\mathrm{d}\mu \qquad (3)$$

and

$$q_l(v) = 2\pi \int [1 - P_l(\cos\theta)]\,\sigma(v,\theta)\sin\theta\,\mathrm{d}\theta . \qquad (4)$$

Here, the symbols have the following meaning: n_0 – neutral particle number density, σ – differential cross-section, θ – deflection angle in the velocity space, T_0 – temperature of the neutral gas, k – Boltzmann's constant, m/m_0 – mass ratio of electrons and neutral particles, P_l – Legendre polynomials.

The inelastic collision contribution $[(\delta f/\delta t)_{\mathrm{in}}]$ neutrals is also described in the Lorentz approximation. The corresponding relation reads

$$\left(\frac{\delta f}{\delta t}\right)_{\mathrm{in}} = \sum_{i<j}\left\{n_i\left[\frac{v_{ij}^2}{v} Q_{ij}(v_{ij}) f^0(z,v_{ij}) - v\,Q_{ij}(v) f(z,v,\mu)\right] + \frac{g_i}{g_j} n_j \left[v\,Q_{ij}(v) f^0(z,v_{ji}) - \frac{v_{ij}^2}{v} Q_{ij}(v_{ij}) f(z,v,\mu)\right]\right\} \qquad (5)$$

with

$$v_{ij}^2 = v^2 + \frac{2}{m}(\varepsilon_j - \varepsilon_i) . \qquad (5\,a)$$

The sum in Eq. (5) is extended over all atomic states i, j with excitation energies ε_i, ε_j and statistical weigths g_i, g_j. n_i designates the number density of particles in the i-th state.

In this collision term we assumed isotropic scattering described by the cross section $Q_{ij}(v)$ for the transition $i \rightarrow j$. Further simplifications are possible in special cases *.

Formulating the contributions of the electron-electron encounters $[(\delta f/\delta t)_{\mathrm{ee}}]$ we recall that we are dealing with moderately ionized systems and may neglect angular scattering by electrons in comparison with that by neutrals. We therefore start with the simplified Fokker-Planck collision term [3]

$$\left(\frac{\delta f}{\delta t}\right)_{\mathrm{ee}} = \frac{4\pi n_{\mathrm{e}} e^4}{m^2} \ln\Lambda \frac{1}{v^2}\frac{\partial}{\partial v}\left[A v \frac{\partial f}{\partial v} + B f\right] \qquad (6)$$

with the scattering coefficients

$$A(z,v) = \frac{4\pi}{3 n_{\mathrm{e}}}\left\{\frac{1}{v^2}\int_0^v u^4 f^0(z,u)\,\mathrm{d}u + v\int_v^{\infty} u\, f^0(z,u)\,\mathrm{d}u\right\} \qquad (6\,a)$$

and

$$B(z,v) = \frac{4\pi}{n_{\mathrm{e}}}\int_0^v u^2 f^0(z,u)\,\mathrm{d}u . \qquad (6\,b)$$

Equation (6) describes the energy exchange due to electron-electron collisions. $\ln\Lambda$ designates the Coulomb logarithm and n_{e} the electron density.

Solution Method

Equation (1) is a nonlinear integro-differential equation which, of course, is in general not open to a closed analytic solution. Consequently, we aim

* For the most part the first and the last term in Eq. (5) (particle gain due to first kind and particle loss due to second kind inelastic collisions) can be omitted. Moreover, it is frequently possible to apply one or another of the following simplifications: 1. Restriction to $i=0$, 2. reduction of the level scheme to one effective level, and, 3. neglect of de-exciting collisions [second line of Equation (5)].

to introduce a procedure which yields an approximate solution, and simultaneously provides the basis for an iteration process.

To prepare this procedure we use a suitable split-up of the collision term into two parts, S_0 and S_1, and write

$$\frac{\mu}{n_0 q}\frac{\partial f}{\partial z} - S_0(v; f) = S_1(z, v, \mu; f)\,, \qquad (7)$$

where q is the velocity independent average value of the momentum transfer cross section $q_1(v)$. The subdivision is chosen such that S_0 is linear in f and does not depend explicitely on the space and angular coordinates z, μ. Further, we require that Eq. (7) with the r.h.s. zero can be solved by separation technique. All remaining freedom in the split up is utilized to represent all contributions to the collision term as good as possible by S_0 only.

To account for the r.h.s. of Eq. (7) we develop an iteration procedure where $S_1(z, v, \mu; f)$ is considered as known from the preceding step. To this end we will construct the Green's function of the l.h.s. of Equation (7).

Along the lines given above, we deal with the various contributions to the collision integral as follows:

To define the part S_0^{el} for the *elastic contribution* to the collision integral we first consider the neutral gas temperature T_0 as constant. Moreover we have to remove the explicite μ-dependence in Equation (2). Rewriting the second term on the r.h.s. in the form

$$-n_0 q v\left\{f - f_0 + \sum_{l=1}^{\infty}\left(\frac{q_l}{q} - 1\right) f^l P_l(\mu)\right\} \qquad (8)$$

we find it obvious to chose

$$S_0^{\mathrm{el}}(f) = \frac{1}{q v^3}\frac{\partial}{\partial v}\left\{\frac{m}{m_0} q_1 v^4\left(f + \frac{k T_0}{m v}\frac{\partial f}{\partial v}\right)\right\} - f + f^0\,. \qquad (9)$$

The best choice of the *inelastic contribution* S_0^{in} depends on the special problem under investigation. Without specifying the details we may write

$$S_0^{\mathrm{in}}(f) = -\,[Q(v)/q]\, f \qquad (10)$$

which represents the second term on the r.h.s. of Eq. (5) if we take

$$Q(v) = \sum_{i<j}(n_i/n_0)\, Q_{ij}(v)\,. \qquad (10\,\mathrm{a})$$

Formulating the *contribution of the electron-electron collisions* S_0^{ee} we have to remove the nonlinearity and the space dependence in accordance with the mathematical restrictions given above. We therefore start in this zeroth approximation with a constant electron temperature T_e and a constant degree of ionization $\alpha = n_e/n_0$, and calculate the coefficients A and B in Eqs. (6 a, b) from the Maxwellian distribution corresponding to T_e and n_e. This yields

$$S_0^{\mathrm{ee}}(f) = \frac{1}{q v^3}\frac{\partial}{\partial v}\left[C(v)\left(f + \frac{k T_e}{m v}\frac{\partial f}{\partial v}\right)\right] \qquad (11)$$

with the coefficient

$$C(v) = \frac{4\pi\alpha e^4}{m^2}\ln\Lambda\left\{\mathrm{erf}(x) - \frac{2}{\pi^{1/2}} x e^{-x^2}\right\};$$
$$x = \sqrt{\frac{m v^2}{2 k T_e}}\,. \qquad (11\,\mathrm{a})$$

We can now collect the contributions (9), (10), (11), and write down the complete term S_0. Using the transformation

$$y = \int_0^v \frac{(m/m_0)\, q_1(v) + C v^{-4}}{(m/m_0)\, q_1(v) + (T_e/T_0)\, C v^{-4}}\frac{m v}{k T_0}\,\mathrm{d}v;$$
$$F(z, y, \mu) = e^y f(z, v, \mu) \qquad (12)$$

to bring the differential part of the operator into the self-adjoint form, we obtain

$$S_0(f) = e^{-y}\left\{\frac{e^y}{g}\frac{\partial}{\partial y}\left(p e^{-y}\frac{\partial F}{\partial y}\right) - \frac{Q}{q}F - F + \tfrac{1}{2}\int_{-1}^{1} F\,\mathrm{d}\mu\right\} \qquad (13)$$

with

$$p(y) = (m/m_0)\, q_1(v) v^4 + C\,|_{v = v(y)} \qquad (13\,\mathrm{a})$$

and

$$g(y) = \left[1 + \frac{(T_e - T_0)\, C}{T_0\, p(y)}\right]\frac{k T_0}{m} q v^2(y)\,. \qquad (13\,\mathrm{b})$$

Solution by Separation

To prepare the solution by separation we introduce the dimensionless space variable

$$s = q\int_0^z n_0(z)\,\mathrm{d}z \qquad (14)$$

and rewrite Eq. (7) with $S_1 = 0$ and Eq. (13) in the form

$$\mu\frac{\partial F}{\partial s} + F = \frac{e^y}{g(y)}\frac{\partial}{\partial y}\left[p(y) e^{-y}\frac{\partial F}{\partial y}\right] - \frac{Q(y)}{q}F + \tfrac{1}{2}\int_{-1}^{1} F\,\mathrm{d}\mu\,. \qquad (15)$$

Using the ansatz

$$F_\nu^\lambda(s,y,\mu) = u_\nu^\lambda(s)\,\varphi_\nu^\lambda(\mu)\,\psi_\lambda(y) \tag{16}$$

with the separation parameters λ and ν, we find

$$u_\nu^\lambda(s) = \exp\left[-\frac{(1+\lambda)\,s}{\nu}\right], \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[p\,e^{-y}\,\frac{\mathrm{d}\psi_\lambda}{\mathrm{d}y}\right] + g\,e^{-y}\left[\lambda - \frac{Q}{q}\right]\psi_\lambda = 0\,, \tag{18}$$

$$(\nu-\mu)\,\varphi_\nu^\lambda(\mu) = \frac{\nu}{2(1+\lambda)}\int_{-1}^{1}\varphi_\nu^\lambda(\mu)\,\mathrm{d}\mu\,. \tag{19}$$

Equation (18) together with the boundary condition that ψ_λ is square-integrable with the weight function $g(y)e^{-y}$ presents an eigenvalue problem of the Sturm type and has a complete system of orthogonal eigenfunctions $\{\psi_{\lambda n}\}$ with discrete eigenvalues $\lambda_n \geqq 0$ *.

The theory of distributions shows [4] that the general solution of Eq. (19) is

$$\varphi_\nu^\lambda(\mu) = \frac{1}{2(1+\lambda)}\,\mathrm{P}\frac{\nu}{\nu-\mu} + \varrho_\lambda(\nu)\,\delta(\nu-\mu) \tag{20}$$

if we prescribe the normalization

$$\int_{-1}^{1}\varphi_\nu^\lambda(\mu)\,\mathrm{d}\mu = 1\,. \tag{20 a}$$

For any value of ν in the range $[-1, 1]$ this normalization can be satisfied if we use

$$\varrho_\lambda(\nu) = 1 - \frac{\nu}{2(1+\lambda)}\log\frac{1+\nu}{1-\nu}\,. \tag{21}$$

That means here we have a continuous spectrum of eigenvalues and eigenfunctions.

In the range $\nu \notin [-1, 1]$, however, Eq. (20) together with (20 a) yields the dispersion relation

$$\Lambda_\lambda(\nu) \equiv 1 - \frac{\nu}{2(1+\lambda)}\log\frac{\nu+1}{\nu-1} = 0\,, \tag{22}$$

which prescribes a discrete set of two eigenvalues $+\nu_\lambda$ and $-\nu_\lambda$ for any value $\lambda > 0$. The corresponding separation solutions read

$$F_{\pm\nu_\lambda}^\lambda(s,y,\mu) = \frac{\nu_\lambda\,e^{\mp(1+\lambda)s/\nu_\lambda}}{2(1+\lambda)\,(\nu_\lambda+\mu)}\,\psi_\lambda(y) =: \Phi_\pm^\lambda(s,\mu)\,\psi_\lambda(y)\ (\lambda>0)\,. \tag{23 a}$$

In the limit $\lambda \to +0$ we have $\psi_\lambda \to$ const and $\nu_\lambda \to \infty$. From the linear combinations $\Phi_+^\lambda \pm \Phi_-^\lambda$ we find in this limit two solutions

$$\Phi_+^0(s,\mu) \equiv 1\,; \quad \Phi_-^0(s,\mu) = s - \mu \tag{23 b}$$

representing the quasi-equilibrium $f = e^{-y}$ and constant diffusion $f^0 = s\,e^{-y}$, $f^1 = e^{-y}$ of the electrons.

Expansion in Terms of the Eigenfunctions

Case and Zweifel [5] have shown that the continuous spectrum together with the discrete one forms a complete set of eigenfunctions. We can therefore perform an expansion of our distribution function in terms of the eigensolutions:

$$F(s,y,\mu) = \sum_{\lambda\,\in\,\{\lambda_i\}}\left[A_+^\lambda\,\Phi_+^\lambda + A_-^\lambda\,\Phi_-^\lambda + \int_{-1}^{1} a_\nu^\lambda\,\varphi_\nu^\lambda(\mu)\,\exp\{-(1+\lambda)\,s/\nu\}\,\mathrm{d}\nu\right]\psi_\lambda(y)\,. \tag{24}$$

The coefficients $A_\pm^\lambda$, a_ν^λ, of course, must be determined from the boundary conditions of our problem. We will assume that the half distribution functions $f(s_-, y, \mu > 0)$ and $f(s_+, y, \mu < 0)$ are prescribed at the positions s_- and s_+ respectively ** $(s_+ - s_- \gg 1)$. In the case $s_+ = \infty$, the second boundary condition is not needed.

Utilizing the orthogonality of the functions ψ_λ, the above boundary conditions produce from Eq. (24) two singular integral equations for the spectral functions a_ν^λ and $a_{-\nu}^\lambda$ $(\nu \geqq 0)$

* Strictly speaking, particle conservation requires that the smallest eigenvalue λ_0 be zero. With $Q>0$, however, we find $\lambda_0>0$. This reflects the fact that we have neglected inelastic particle gain in S_0.

** In the case of completely absorbing or diffusely reflecting walls these half distributions are directly prescribed by the physical boundary conditions. For more general reflection mechanisms, of course, the half distributions $f(s_\pm, y, \mu \lessgtr 0)$ are affected by the unknown distribution functions $f(s_\pm, y, -\mu)$ which have to be evaluated from the analysis. In these cases, consequently, an iteration procedure must be applied.

$$a^{\lambda}_{\pm\mu}\,\varrho_\lambda(\mu)\exp\{\mp(1+\lambda)s_\mp/\mu\}+\frac{1}{2(1+\lambda)}\int_0^1 \mathcal{P}\,\frac{\nu}{\nu-\mu}\exp\{\mp(1+\lambda)s_\mp/\nu\}\,a^{\lambda}_{\pm\nu}\,\mathrm{d}\nu=h^{\lambda}_{\pm} \tag{25}$$

where we have used the abbreviation

$$h^{\lambda}_{\pm}=\int_0^\infty F(s_\pm,y,\pm\mu)\,\psi_\lambda(y)\;g(y)\,e^{-y}\,\mathrm{d}y\;-A^{\lambda}_{+}\Phi^{\lambda}_{+}(s_\mp,\pm\mu)-A^{\lambda}_{-}\Phi^{\lambda}_{-}(s_\mp,\pm\mu)-$$
$$-\int\varphi^{\lambda}_{-\nu}(\mu)\exp\{\pm(1+\lambda)s/\nu\}\,a^{\lambda}_{\mp\nu}\,\mathrm{d}\nu\,. \tag{26 a}$$

The two Eq. (25) are coupled via the last term of Equation (26 a). The coupling coefficient can be shown to be of order $\exp\{-(1+\lambda)(s_+-s_-)/\nu\}$. With $\lambda\geqq 0$ and $\nu\leqq 1$, it follows that this term can be omitted due to our restriction to wall distances $s_+-s_-\gg 1$ *. We therefore have

$$h^{\lambda}_{\pm}=\int_0^\infty F(s_\mp,y,\pm\mu)\,\psi_\lambda(y)\,g(y)\,e^{-y}\,\mathrm{d}y\;-A^{\lambda}_{+}\Phi^{\lambda}_{+}(s_\mp,\pm\mu)-A^{\lambda}_{-}\Phi^{\lambda}_{-}(s_\mp,\pm\mu)\,. \tag{26}$$

The singular integral equation has solutions only if certain conditions are fulfilled. These conditions serve to specify the coefficients $A^{\lambda}_{\pm}$.

With respect to the derivation of the conditions of solubility, as well as the solution of the singular integral equation, we refer the reader to the work [5] which is directly applicable to our present problem. From this work we find

$$a^{\lambda}_{\pm\nu}=\exp\left\{\pm\frac{\lambda+1}{\nu}\,s_\mp\right\}\frac{H_\lambda(\nu)}{N_\lambda(\nu)}\int_0^1\frac{\mu}{H_\lambda(\mu)}\;h^{\lambda}_{\pm}\varphi_\nu{}^{\lambda}(\mu)\,\mathrm{d}\mu \tag{27}$$

with

$$N_\lambda(\nu)=\nu\left\{\varrho_\lambda{}^2(\nu)+\left[\frac{\pi\nu}{2(1+\lambda)}\right]^2\right\} \tag{27 a}$$

and

$$H_\lambda(\nu)=\frac{(\nu_\lambda{}^2-\nu^2)\lambda}{(1+\nu)(1+\lambda)}\exp\left\{\frac{1}{\pi}\int_0^1\frac{\arg\Lambda^{+}_{\lambda}(\nu')}{\nu+\nu'}\,\mathrm{d}\nu'\right\} \tag{27 b}$$

and

$$\arg\Lambda^{+}_{\lambda}(\nu)=\mathrm{atn}\,\frac{\pi\nu}{2(1+\lambda)+\nu\log\dfrac{1-\nu}{1+\nu}}\,. \tag{27 c}$$

The coefficients $A^{\lambda}_{\pm}$ can be determined from the conditions

$$\int_0^1[\mu/H_\lambda(\mu)]\,h^{\lambda}_{\pm}\,\mathrm{d}\mu=0\,. \tag{28}$$

* This means that the distribution function at the first wall is not affected by details of the angular distribution at the second wall and vice versa. Note that we do not neglect the coupling via the coefficients $A^{\lambda}_{\pm}$ describing, e.g., particle flux and heat conduction from wall to wall.

The Green's Function

To give the basis for the iteration procedure, we finally have to construct the Green's function $G(s,y,\mu\,|\,s',y',\mu')$ which is defined through the equation

$$\mu(\partial G/\partial s-S_0(G)=\delta(s-s')\,\delta(y-y')\,\delta(\mu-\mu') \tag{29}$$

with the boundary conditions

$$G(s_-\ldots|\ldots)=G(s_+\ldots|\ldots)=0\,. \tag{29 a}$$

In terms of Green's function the electron distribution is given by

$$f(s,y,\mu)=f_{\mathrm{h}}(s,y,\mu)+\int S_1(z',y',\mu')\,G\,\mathrm{d}z'\,\mathrm{d}y'\,\mathrm{d}\mu' \tag{30}$$

where f_{h} designates the zeroth approximation (i. e. the solution of Eq. (7) with the r.h.s. put to zero) given above.

To reduce our task to the more convenient problem of finding the Green's function G_0 for an infinite medium we write

$$G=G_0-G_1 \tag{31}$$

where G_0 satisfies Eq. (29) with the boundary conditions

$$G_0(\pm\infty,\ldots|\ldots)=0\,. \tag{29 b}$$

Knowing G_0 we can find G_1 from the homogeneous equation

$$\mu\frac{\partial G_1}{\partial s}-S_0(G_1)=0 \tag{32}$$

with the boundary conditions

$$G_1(s_\pm,\ldots|\ldots)=G_0(s_\pm,\ldots|\ldots) \tag{32 a}$$

by the same formalism given above to find f_{h}.

To determine G_0, we write

$$G_0 = \sum_\lambda g_\lambda(s,\mu\,|\,s',\mu')\,\psi_\lambda(y)\,\psi_\lambda(y')\,g(y)\,e^{-y} \quad (33)$$

and

$$g_\lambda = \begin{cases} B_+^\lambda\,\Phi_+^\lambda(s,\mu) + \int_0^1 b_\nu^\lambda\,\varphi_\nu^\lambda(\mu)\,e^{-(1+\lambda)s/\nu}\,\mathrm{d}\nu & (s>s') \\ -B_-^\lambda\,\Phi_-^\lambda(s,\mu) + \int_0^1 b_\nu^\lambda\,\varphi_\nu^\lambda(\mu)\,e^{-(1+\lambda)s/\nu}\,\mathrm{d}\nu & (s<s') \end{cases} \quad (34)$$

where we have used the boundary condition (29 b). From the jump condition

$$G_0 = (s'+0,\ldots\,|\ldots) - G_0(s'-0,\ldots\,|\ldots) = [\delta(\mu-\mu')/\mu]\,\delta(y-y') \quad (35)$$

which follows by integration of Eq. (29) and from the completeness relation

$$\delta(y-y') = \sum_\lambda \psi_\lambda(y)\,\psi_\lambda(y')\,g(y)\,e^{-y} \quad (36)$$

we get

$$g_\lambda(s'+0\ldots\,|\ldots) - g_\lambda(s'-0\ldots\,|\ldots) = \delta(\mu-\mu')/\mu\,. \quad (37)$$

The solution of Eq. (34) together with (37) again may be taken from [5].

The result is

$$g_\lambda(s,\mu\,|\,s',\mu') = \frac{\Phi_\pm^\lambda(s,\mu)\,\Phi_\pm^\lambda(s',\mu')}{\frac{\nu_\lambda^3}{2(1+\lambda)}\left[\frac{1}{(1+\lambda)(\nu_\lambda^2-1)} - \frac{1}{\nu_\lambda^2}\right]} + \int_0^1 \frac{\varphi_{\pm\nu}^\lambda(\mu)\,\varphi_{\pm\nu}^\lambda(\mu')\exp\{-(1+\lambda)\,|s-s'|/\nu\}}{N_\lambda(\nu)}\,\mathrm{d}\nu \quad \text{for } s \gtrless s'. \quad (38)$$

With Eq, (31), (32), (33) and (38) the problem of finding Green's function is solved in the most general case. It should be noted, however, that simplifications are possible for many applications. Particularly, it is sufficient to take only G_0 for the description of the energy relaxation if angular and energy relaxation occur on different scales. This approximation leads to the correct contribution of inelastic and electron-electron collisions, and can account for the effect of a weak electric field.

Numerical Evaluation

We demonstrate here the application of the general procedure outlined above. In this first approach we aim to be transparent and instructive, and for these reasons we limit our evaluation to the case $S_1 = 0$. Further, we assume $T_e = T_0$, $q_1(v) \equiv q$ and $Q(v) \equiv 0$.

We return to Eq. (18), which under these assumptions reads (cf. Eqs. 11 a, 12, 13 a, 13 b)

$$\frac{\mathrm{d}}{\mathrm{d}y}\left\{\left[y^2 + \gamma\left(\mathrm{erf}(y^{1/2}) - \frac{2}{\pi^{1/2}}\,y^{1/2}\,e^{-y}\right)\right]e^{-y}\,\frac{\mathrm{d}\psi_\lambda}{\mathrm{d}y}\right\} + \frac{m_0}{2m}\,\lambda\,y\,e^{-y}\,\psi_\lambda(y) = 0 \quad (39)$$

with

$$y = m\,v^2/2\,k\,T_0 \quad (39\,a)$$

and

$$\gamma = \frac{\pi\,a}{q\,m/m_0}\left(\frac{e^2}{k\,T_0}\right)^2 \ln\Lambda\,. \quad (39\,b)$$

The parameter γ characterizes the energy exchange due to electron-electron collisions compared to the energy exchange due to electron-neutral interaction. If this parameter is zero, Eq. (39) is the Laguerre differential equation (see e. g. [6]) and the eigensolutions are Laguerre polynomials of the first kind L_n^1 with the eigenvalues

$$(m_0/2\,m)\,\lambda_n = n = 0, 1, 2, \ldots\,. \quad (40)$$

Neclecting terms of higher order in $(n\,m/m_0)$, Eq. (28) yields for the discrete eigenvalues

$$\nu_{\lambda_n} = \sqrt{1/3\,\lambda_n} = \sqrt{m_0/6\,n\,m}\,. \quad (41)$$

These large relaxation lengths demonstrate how ineffective the energy exchange of the electrons with the neutrals is.

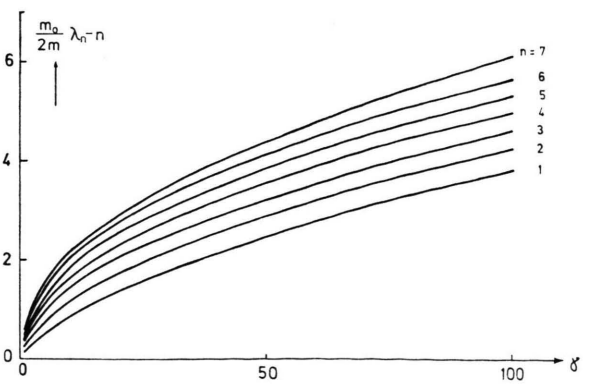

Fig. 1. Dependence of the eigenvalues λ_n on the relative electron-electron collision frequency γ [cf. Equation (39b)].

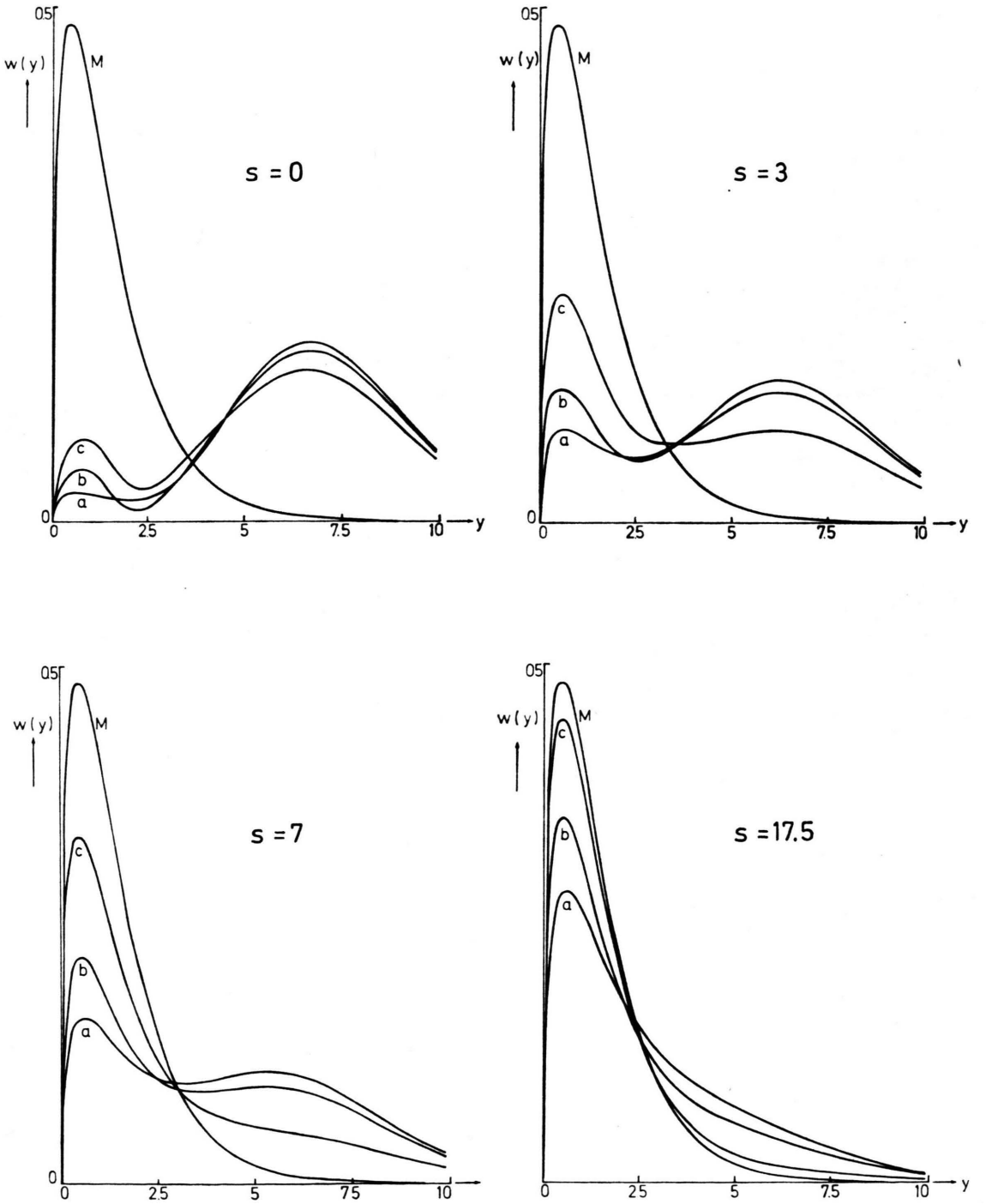

Fig. 2. Energy distribution $w(z, y) = \text{const}\, y^{1/2} \int f(z, y, \mu)\, d\mu$ for various values of the wall distance s and the relative electron-electron collision frequency γ [a) $\gamma=0$; b) $\gamma=10$; c) $\gamma=100$]. The curve M gives the Maxwell distribution.

In the case $\gamma \neq 0$ Eq. (39) cannot be solved in closed form. We calculated the eigenfunctions numerically by utilizing the Ritz procedure.

Figure 1 demonstrates the variation of the eigenvalue spectrum as a function of the parameter γ.

For the numerical evaluation of the distribution function, we need, of course, a specification of the boundary conditions. We choose the example of an electron beam with a thermal velocity spread

$$f(0, y, \mu) = \delta(\mu - 1) \exp\left\{-\left(\frac{y - y_0}{a}\right)^2\right\} \quad (42)$$

emitted from an absorbing wall at $s_- = 0$ into an unbounded plasma ($s_+ = \infty$). Our figures refer to parameter values $y = 6.5$ and $a = 3$.

The evaluations shown in Fig. 2 demonstrate the dependence of the distribution function on the influence of the electron-electron collisions (γ) as well as the dependence on the distance from the emitting wall (s). For the purpose of comparison, the curve (M) describes the Maxwellian distribution.

The curves clearly exhibit the Maxwellisation with increasing distance from the emitting wall and with increasing influence of the electron collisions. In the figure $s = 0$ the distinction of the orginal beam from the contribution of the scattered particles is most pronounced. The beam corresponds to the large peak at $y = 6.5$ whereas the small peak near $y = 1$ is due to the scattering process in the plasma. It is also reflected in the figures that, due to the decreasing cross section for high particle velocities, the Maxwellisation process for large values of the energy y is less effective than for small values.

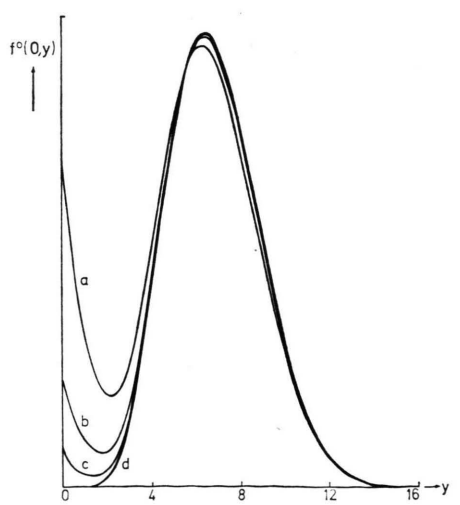

Fig. 3. Distribution function $f^0(0, y) = \frac{1}{2} \int f(0, y, \mu)\, d\mu$ at the wall for $\gamma = 0$ and various values of the atomic weight A [a) $A = 1$; b) $A = 10$; c) $A = 100$].

The influence of the atomic weight $A = m_0/m_p$ on the Maxwellisation is demonstrated in Figure 3. Here we have shown the distribution function at the wall for four values of the mass ratio. The curve d describes the distribution of the injected beam. The increase in the energy loss with decreasing mass of the plasma is quite conspicuous.

Due to the angular integrations the preceding curves do not yield information about the development of the angular distribution. Figure 4 aims to demonstrate this development. It gives for six values of the distance (s) a presentation of the distribution function in polar coordinates. For given values of the particle energy we have plotted in the radial direction the value of the distribution function. The heavy line indicates the direction of the original beam. A comparison of Fig. 2 shows how the angular relaxation process occurs much faster than the Maxwellisation.

Summary

We describe a method to simultaneously calculate the angular- and energy relaxations of electrons emitted from a wall into the plasma. A suitable subdivision of the collision term transforms the problem into two parts: The solution of a zero order eigenvalue problem and an iteration procedure accounting for the rest of the collision term. Already the zeroth order solution is suitable to give a reasonable approximation for elastic collisions with neutrals and electrons and for the particle loss due to inelastic collisions.

The zeroth order problem is essentially reduced to an eigenvalue problem of the Sturm type. The coefficients of the eigenfunction expansion can be calculated from known techniques of singular integral equations. To prepare the iteration procedure we constructed the Green's function of the zeroth order equation.

For an instructive case we evaluated the formalism. The results describe in all detail quantitatively the complicated interplay of the angular- and energy relaxation in general agreement with physical expectations. We plan to further apply this method in cases where – due to inelastic collisions – the

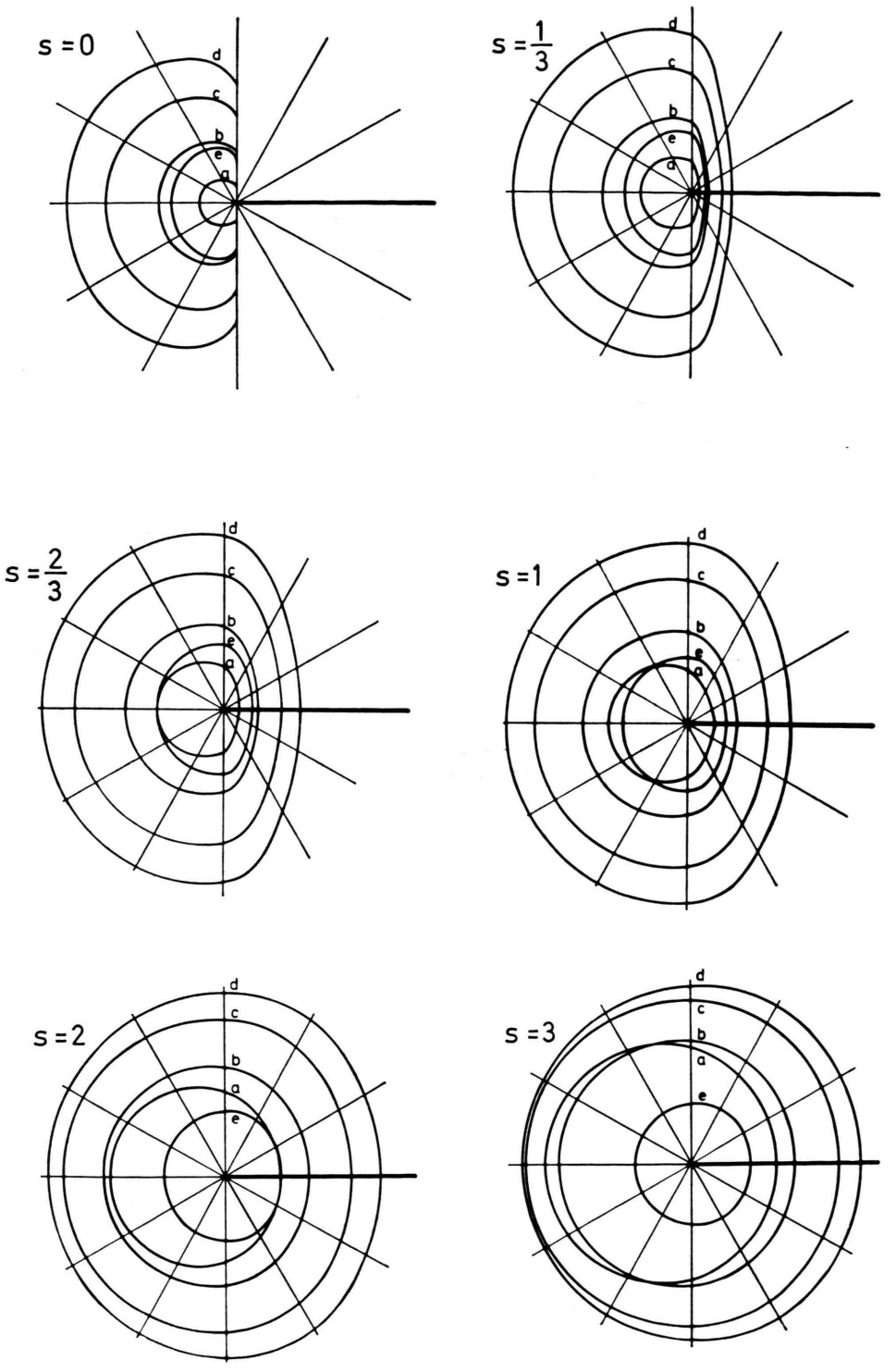

Fig. 4. Angular relaxation of the distribution function for $\gamma=0$ in polar coordinates $r(\theta)=f(s, y, \cos\theta)$. The curves a, b, c, d, e refer to energies $y=1$, 4, 5, 7 and 10 respectively.

angular and the energy relaxation occur on the same scale. Also it is possible to account for additional terms the kinetic equation (particularly an electric field) via the Green's function.

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